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Effect Test Spaces

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An effect test space, or E-test space, for short, is a generalization of a test space that is able to describe unsharp measurements. Effects in an E-test space correspond to yes–no measurements, and observables correspond to general measurements that may have more than two values. Sharpness, compatibility, and orthogonality of effects are considered. It is shown that every observable is determined by its eigenvalues and eigeneffects. The spectrum of an observable is studied and special types of observables are investigated. Orthocomplements and a natural local sum on an E-test space are introduced. Relationships between the resulting structures and previously studied frameworks are presented.

1. INTRODUCTION

In 1972, Foulis and Randall (also see Randall and Foulis, 1973) introduced test spaces (or quasimanuals) as a basis of a language for the empirical sciences. As emphasized by Foulis and Randall, test spaces give a direct description of laboratory operations. Information is lost in moving from a test space to its corresponding logic, and for this reason, test spaces provide a more fundamental description of a physical system. Twenty-two years later, Dvurečenskij and Pulmannová (1994; also see Pulmannová and Wilce, 1995) presented a generalization of test spaces called D-test spaces in order to include a description of unsharp measurements. Subsequently, the author studied an equivalent framework called an effect test space, or E-test space, for short (Gudder, n.d.).

It is shown in Dvurečenskij and Pulmannová (1994) and Gudder (n.d.) that a certain type of E-test space called an algebraic E-test space can be organized into an effect algebra (Bennett and Foulis, 1995; Dalla Chiara and Giuntini, 1994; Dvurečenskij, 1995; Foulis and Bennett, 1994; Giuntini and

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Greuling, 1989; Greechie and Foulis, 1995; Kôpka, 1992; Kôpka and Chovanec, 1994) in a natural way. This effect algebra $\Pi(X)$ corresponds to the logic of the E-test space X. As before, information is lost in moving from X to $\Pi(X)$, and X provides a more fundamental description of a physical system. Moreover, the algebraic condition does not appear to have a clear physical motivation, and for this reason the logic may not even exist. In this paper we shall study E-test spaces in their own right and we shall not be concerned with their relationships to effect algebras.

We begin by motivating the concept of an E-test space and presenting some basic definitions. The most important of these definitions are the properties of sharpness, compatibility, and orthogonality for effects in an E-test space. Although an effect corresponds to a yes–no measurement, an observable corresponds to a general measurement that can have more than two values. We show that every observable is determined by its eigenvalues and eigeneffects. The spectrum of an observable is studied and various spectral mapping theorems are proved. Special types of observables called universal and maximal observables are defined and investigated.

Two kinds of orthocomplements on an E-test space are introduced and their properties are studied. The orthocomplements are local in the sense that they depend on the test being performed. A natural local sum and other operations depending on this sum are defined. Relationships between the resulting structures, MV-algebras (Cattaneo *et al.*, n.d.; Chang, 1957, 1958; Dalla Chlara and Giuntini, 1994) quantum MV-algebras (Giuntini, 1995, n.d.), and BZ-algebras (Cattaneo, 1993; Cattaneo and Giuntini, 1995; Cattaneo and Nisticò, 1989) are discussed. Finally a Sasaki mapping on an E-test space is considered and its connection to classical and semi-classical E-test spaces is demonstrated (Bennett and Foulis, 1995).

2. MOTIVATION AND DEFINITIONS

We begin with some motivation that underlies our definition of an effect test space. We consider a test t to be a perfectly accurate (sharp) yes–no measurement. Although a test is an idealization that cannot be attained in practice, experimentalists continually refine their measuring instruments to closely approximate a test. Let x be a possible outcome of an experiment that is relevant to (testable by) t. If x occurs, then t registers yes, and if x does not occur, t registers no. For example, a particle detector t tests whether a particle is in a certain region $A \subseteq \mathbb{R}^3$. If a particle is at a point $x \in A$, then t clicks, and otherwise, t does not click.

Now make N runs of an experiment, where N is a large integer. Suppose that the outcomes $S(t) = \{x_1 \ldots, x_n\}$ that are testable by t are among the possible outcomes of the experiment. Let $t(x_i)$ be the number of times that

x_i occurs and hence are ideally registered as a yes by t, $i = 1, \ldots, n$. Then we can consider t as a function $t: S(t) \to \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We think of an effect f as a submeasurement of t that may not be perfectly accurate (can be unsharp). Thus, effects correspond to yes–no measurements that are actually realizable. Let $S(f) = \{x_1, \ldots, x_m\}$, $m \leq n$, be the set of outcomes that are testable by f. Now rerun the experiment N times and let $f(x_i)$ be the number of times that f registers a yes when x_i occurs. If f is unsharp, then f may not register a yes when $x_i \in S(f)$ occurs. Also, on rare occasions, f may register a false yes for $x_i \in S(f)$ even when x_i does not occur. However, on the average we would have $f(x_i) \leq t(x_i)$, $i = 1, \ldots, m$. Notice that the effect f is sharp if only if $f(x_i) = t(x_i)$ for all $x_i \in S(f)$.

We now present some definitions that are motivated by our previous discussion. Let X be a nonempty set that we call an *outcome space*. The elements of X correspond to the various outcomes of experiments that can be performed on a fixed physical system. We call a function $f \in \mathbb{N}_0^X$ a *multiplicity function* and such functions take account of the fact that an outcome may occur with a certain multiplicity when an experiment is performed many times. For $f, g \in \mathbb{N}_0^X$ we write $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$ and in this case we define $g - f \in \mathbb{N}_0^X$ by $(g - f)(x) = g(x) - f(x)$. For $f, g \in \mathbb{N}_0^X$, we define $f + g \in \mathbb{N}_0^X$ by $(f + g)(x) = f(x) + g(x)$, $f \vee g \in \mathbb{N}_0^X$ by $(f \vee g)(x) = \max(f(x), g(x))$, and $f \wedge g \in \mathbb{N}_0^X$ by $(f \wedge g)(x) = \min(f(x), g(x))$.

If $\mathcal{T} \subseteq \mathbb{N}_0^X$, we call $(X, \mathcal{T})$ an *effect test space* or an *E-test space*, for short, if the following conditions hold.

(E1) For any $x \in X$ there exists a $t \in \mathcal{T}$ such that $t(x) \neq 0$.
(E2) If $s, t \in \mathcal{T}$ with $s \leq t$, then $s = t$.

The elements of $\mathcal{T}$ are called *tests* and $f \in \mathbb{N}_0^X$ is called an *effect* if $f \leq t$ for some $t \in \mathcal{T}$. Condition (E1) states that every outcome occurs for some test and Condition (E2) states that two different tests are not redundant.

Denote by $f_0 \in \mathbb{N}_0^X$ the function $f_0(x) = 0$ for all $x \in X$. By (E1) there exists a $t \in \mathcal{T}$ such that $t \neq f_0$, and since $f_0 \leq t$, it follows from (E2) that $f_0 \notin \mathcal{T}$. We denote the set of effects by $\mathcal{E}(X, \mathcal{T})$. We also use the notation $\mathcal{E}(X)$ or $\mathcal{E}$ if $\mathcal{T}$ or $(X, \mathcal{T})$ are understood. It is clear that $f_0 \in \mathcal{E}$ and $f \wedge g \in \mathcal{E}$ whenever $f, g \in \mathcal{E}$. However, $f \vee g$ need not be in $\mathcal{E}$ for $f, g \in \mathcal{E}$. For $f \in \mathcal{E}$, we define

$$S(f) = \{x \in X: f(x) \neq 0\}$$

An effect f is *sharp* in $t \in \mathcal{T}$ if $f \leq t$ and $f(x) = t(x)$ for all $x \in S(f)$ and f is *globally sharp* if for every $t \in \mathcal{T}$, $f \wedge t$ is sharp in t. Notice that if f is globally sharp and $f \leq t$, then f is sharp in t.

For $t \in \mathcal{T}$, we define $\mathcal{E}_t = \{f \in \mathcal{E}: f \leq t\}$. An indexed family $f_\alpha \in \mathcal{E}$, $\alpha \in A$, is *compatible* if $f_\alpha \in \mathcal{E}_t$, $\alpha \in A$, for some $t \in \mathcal{T}$. If f_α, $\alpha \in A$, are compatible, then they can be performed simultaneously by employing a single test. If f and g are compatible, we write $f \leftrightarrow g$. Notice that $f_0 \leftrightarrow f$ for every $f \in \mathcal{E}$ and that $\leftrightarrow$ is a symmetric and reflexive relation. An indexed family $f_\alpha \in \mathcal{E}$, $\alpha \in A$, is *orthogonal* if $\Sigma f_\alpha \leq t$ for some $t \in \mathcal{T}$. It is clear that if $(f_\alpha: \alpha \in A)$ is orthogonal, then it is compatible. For $f \in \mathcal{E}$, we denote $f + \cdots + f$ (n summands) by nf. We denote the characteristic function of $Y \subseteq X$ by I_Y and use the notation $I_x = I_{\{x\}}$.

Lemma 2.1. (a) For $f \in \mathcal{E}$, $t \in \mathcal{T}$, we have $f \leftrightarrow t$ if and only if $f \leq t$. (b) For $s, t \in \mathcal{T}$, we have $s \leftrightarrow t$ if and only if $s = t$. (c) If $f \leftrightarrow g$ and $f_1 \leq f$, $g_1 \leq g$, then $f_1 \leftrightarrow g_1$.

Proof. (a) If $f \leq t$, then $f, t \in \mathcal{E}_t$, so $f \leftrightarrow t$. Conversely, if $f \leftrightarrow t$, then there exists an $s \in \mathcal{T}$ such that $f, t \leq s$. By (E2), $s = t$, so $f \leq t$. (b) This follows from (E2) and (a). (c) Since $f \leftrightarrow g$, we have $f, g \leq t$ for some $t \in \mathcal{T}$. Hence, $f_1, g_1 \leq t$, so $f_1 \leftrightarrow g_1$. ∎

Theorem 2.2. The following statements are equivalent. (a) $(f_\alpha, \alpha \in A)$ is compatible. (b) $\vee f_\alpha \leq t$ for some $t \in \mathcal{T}$. (c) There exists an orthogonal family $(g_\beta, \beta \in B)$ such that $f_\alpha = \Sigma(g_\beta, \beta \in B_\alpha \subseteq B)$ for every $\alpha \in A$.

Proof. (a) ⇒ (b) If $(f_\alpha, \alpha \in A)$ is compatible, then $f_\alpha \leq t$, $\alpha \in A$, for some $t \in \mathcal{T}$. Hence, $\vee f_\alpha \leq t$. (b) ⇒ (c) Suppose $\vee f_\alpha \leq t$ for $t \in \mathcal{T}$. Since $t = \Sigma(t(x)I_x, x \in S(t))$, we conclude that $(t(x)I_x, x \in S(t))$ is orthogonal. Letting

$$B = \bigcup_{x \in S(t)} \{\beta_1^x, \ldots, \beta_n^x: n = t(x), \beta_i^x = x, i = 1, \ldots, n\}$$

then $(I_\beta: \beta \in B)$ is orthogonal. For any $\alpha \in A$, letting

$$B_\alpha = \bigcup_{x \in S(f_\alpha)} \{\beta_1^x, \ldots, \beta_m^x: m = f_\alpha(x)\} \subseteq B$$

we have that

$$f_\alpha = \sum (I_\beta, \beta \in B_\alpha \subseteq B)$$

(c) ⇒ (a) Since $(g_\beta: \beta \in B)$ is orthogonal, there exists a $t \in \mathcal{T}$ such that $\Sigma g_\beta \leq t$. Then for any $\alpha \in A$ we have

$$f_\alpha = \sum (g_\beta, \beta \in B_\alpha \subseteq B) \leq \sum g_\beta \leq t$$

so $(f_\alpha, \alpha \in A)$ is compatible. ∎

3. OBSERVABLES

We have seen that an effect corresponds to a yes–no measurement that may be unsharp. We now consider general measurements that can have more than two values. The Borel σ-algebra on $\mathbb{R}$ is denoted by $\mathcal{B}(\mathbb{R})$. An observable on $(X, \mathcal{T})$ is a mapping $F: \mathcal{B}(\mathbb{R}) \to \mathcal{E}$ that satisfies:

(O1) $F(\mathbb{R}) \in \mathcal{T}$.
(O2) If $A_i \in \mathcal{B}(\mathbb{R})$, $i \in \mathbb{N}$, are mutually disjoint, then $F(\cup A_i) = \Sigma F(A_i)$, where the summation converges pointwise.

Thus, an observable can be thought of as a collection of effects that satisfies the regularity conditions (O1) and (O2). For $A \in \mathcal{B}(\mathbb{R})$, we interpret $F(A)$ as the effect that is observed when F has a value in A. Hence, $F(A)$ corresponds to the yes–no question: "Does F have a value in A when the measurement F is performed?"

Lemma 3.1. Let F be an observable on $(X, \mathcal{T})$ such that $F(\mathbb{R}) = t$. (a) $F(\emptyset) = f_0$. (b) If $A \subseteq B$, then $F(A) \leq F(B)$ and $F(B) = F(A) + F(B \backslash A)$. (c) For every $A \in \mathcal{B}(\mathbb{R})$, $F(A) \in \mathcal{E}_t$ so $(F(A): A \in \mathcal{B}(\mathbb{R}))$ is compatible. (d) If A_i, $i \in \mathbb{N}$, are mutually disjoint, then $(F(A_i), i \in \mathbb{N})$ is orthogonal. (e) $F(A \cup B) = F(A) + F(B) - F(A \cap B)$. (f) If $A_i \in \mathcal{B}(\mathbb{R})$ is a decreasing (increasing) sequence, then $\lim F(A_i) = F(\cap A_i)$ $(F(\cup A_i))$ pointwise.

Proof. (a) Since

$$F(\emptyset) + F(\mathbb{R}) = F(\emptyset \cup \mathbb{R}) = F(\mathbb{R})$$

we conclude that $F(\emptyset) = f_0$. (b) Since $B = A \cup (B \backslash A)$ and $A \cap (B \backslash A) = \emptyset$, we have $F(B) = F(A) + F(B \backslash A)$ and $F(A) \leq F(B)$. (c) Applying (b), we have $F(A) \leq F(\mathbb{R}) = t$ for every $A \in \mathcal{B}(\mathbb{R})$. (d) Since

$$\sum F(A_i) = F(\cup A_i) \leq t$$

we conclude that $(F(A_i), i \in \mathbb{N})$ is orthogonal. (e) Since

$$A \cup B = (A \setminus A \cap B) \cup (B \setminus A \cap B) \cup (A \cap B)$$

and the terms on the right side are mutually disjoint, it follows from (b) that

$$F(A \cup B) = F(A \backslash A \cap B) + F(B \backslash A \cap B) + F(A \cap B)$$
$$= F(A) + F(B) - F(A \cap B)$$

(f) For $x \in S(t)$, define $m: \mathcal{B}(\mathbb{R}) \to \mathbb{R}$ by $m(A) = F(A)(x)$. Then $0 \leq m(A) \leq t(x)$ and if $A_i \in \mathcal{B}(\mathbb{R})$ are mutually disjoint, we have

$$m(\cup A_i) = F(\cup A_i)(x) = \sum F(A_i)(x) = \sum m(A_i)$$

Hence, m is a bounded measure on $\mathcal{B}(\mathbb{R})$. If $A_i \in \mathcal{B}(\mathbb{R})$ is a decreasing sequence, it follows from the monotone convergence theorem for bounded measures that

$$\lim F(A_i)(x) = \lim m(A_i) = m(\cap A_i) = F(\cap A_i)(x)$$

A similar result holds for increasing sequences. ■

Two observables F and G are compatible (written $F \leftrightarrow G$) if $F(\mathbb{R}) = G(\mathbb{R})$. Notice that $\leftrightarrow$ is an equivalence relation for observables. If F is an observable and $u: \mathbb{R} \to \mathbb{R}$ is a Borel function, we define the observable $u(F)$: $\mathcal{B}(\mathbb{R}) \to \mathcal{E}$ by $u(F)(A) = F(u^{-1}(A))$.

Lemma 3.2. (a) $F \leftrightarrow G$ if and only if $F(A) \leftrightarrow F(B)$ for all $A, B \in \mathcal{B}(\mathbb{R})$. (b) If $F = u(H)$, $G = v(H)$, then $F \leftrightarrow G$, $F \leftrightarrow H$, $G \leftrightarrow H$.

Proof. (a) If $F \leftrightarrow G$, then by Lemma 3.1(b) we have

$$F(A), G(B) \leq F(\mathbb{R}) \in \mathcal{T}$$

so that $F(A) \leftrightarrow G(B)$. Conversely, if $F(A) \leftrightarrow G(B)$ for every $A, B \in \mathcal{B}(\mathbb{R})$, then $F(\mathbb{R}) \leftrightarrow G(\mathbb{R})$. Applying Lemma 2.1(b), we have $F(\mathbb{R}) = G(\mathbb{R})$, so that $F \leftrightarrow G$. (b) Since

$$F(\mathbb{R}) = H(u^{-1}(\mathbb{R})) = H(\mathbb{R}) = H(v^{-1}(\mathbb{R})) = G(\mathbb{R})$$

the result follows. ■

If $F(\{\lambda\}) \neq f_0$, then λ is an *eigenvalue* of F and $F(\{\lambda\})$ is the corresponding *eigeneffect*. An eigenvalue is a value that F can attain and the corresponding eigeneffect is observed when F has that value. We denote the set of eigenvalues of F by $\sigma_p(F)$ and call $\sigma_p(F)$ the *point spectrum* of F. In general, $\sigma_p(F)$ need not be a Borel set. For example, let $X \subseteq \mathbb{R}$ be a nonmeasurable set and let $\mathcal{T} = \{I_X\}$. Define $F: \mathcal{B}(\mathbb{R}) \to \mathcal{E}(X, \mathcal{T})$ by $F(A) = I_{A \cap X}$. Then F is an observable with $\sigma_p(F) = X$. The same procedure shows that any subset of $\mathbb{R}$ is the point spectrum of some observable. The next result shows that an observable is determined by its eigenvalues and eigeneffects. An observable F is *atomic* if for every $A \in \mathcal{B}(\mathbb{R})$ we have

$$F(A) = \sum (F(\{\lambda\}): \lambda \in A \cap \sigma_p(F)) \tag{3.1}$$

Theorem 3.3. Every observable is atomic.

Proof. Let F be an observable on $(X, \mathcal{T})$ with $F(\mathbb{R}) = t \in \mathcal{T}$. Let $[a_1, b_1] \subseteq \mathbb{R}$ be a closed interval, $x \in X$, and suppose that $F([a_1, b_1])(x) \neq 0$. If c_1 is the midpoint of $[a_1, b_1]$, then either $F([a_1, c_1])(x) \neq 0$ or $F([c_1, b_1])(x) \neq 0$. Continuing this process, we obtain a decreasing sequence of closed intervals $[a_i, b_i]$ such that $\lim (b_i - a_i) = 0$ and $F([a_i, b_i])(x) \geq 1$, $i = 1, 2, \ldots$. Since $\mathbb{R}$ is complete, $\cap [a_i, b_i] = \lambda \in \mathbb{R}$, and it follows from Lemma 3.1(f) that

$$F(\{\lambda\})(x) = \lim F([a_i, b_i])(x) \geq 1$$

If $A \in \mathcal{B}(\mathbb{R})$ has the property that $F(A)(x) \neq 0$ implies that there exists a $\lambda \in A$ such that $F(\{\lambda\})(x) \neq 0$, we say that A is *F-atomic*. We have thus shown that any closed interval is F-atomic. Since $\mathbb{R} = \cup[i, i + 1]$, $i \in \mathbb{Z}$, it follows that $\mathbb{R}$ is F-atomic. Since any open interval $(a, b) \subseteq \mathbb{R}$ is the union of an increasing sequence of closed intervals, it follows from Lemma 3.1(f) that (a, b) is F-atomic. If $x \in S(t)$, then $F(\mathbb{R})(x) = t(x) \neq 0$. Hence, there exists a $\lambda_1 \in \mathbb{R}$ such that $F(\{\lambda_1\})(x) \neq 0$. If $F(\{\lambda_1\})(x) \neq t(x)$, then

$$F(\mathbb{R}\backslash\{\lambda_1\})(x) = t(x) - F(\{\lambda_1\})(x) \neq 0$$

Since $\mathbb{R}\backslash\{\lambda_1\}$ is the union of two open intervals, there exists a $\lambda_2 \in \mathbb{R}\backslash\{\lambda_1\}$ such that $F(\{\lambda_2\})(x) \neq 0$. If

$$F(\{\lambda_1\})(x) + F(\{\lambda_2\})(x) \neq t(x)$$

this same procedure gives a $\lambda_3 \in \mathbb{R}\backslash\{\lambda_1, \lambda_2\}$ such that $F(\{\lambda_3\})(x) \neq 0$. Continuing this process, there exists a finite set

$$S_x = \{\lambda_1, \ldots, \lambda_n\} \subseteq \mathbb{R}$$

such that $F(\{\lambda_i\})(x) \neq 0$, $i = 1, \ldots, n$, and $\Sigma F(\{\lambda_i\})(x) = t(x)$.

If $A \in \mathcal{B}(\mathbb{R})$, we have

$$F(A)(x) \geq \sum (F(\{\lambda\})(x): \lambda \in A \cap S_x)$$

Suppose that

$$F(A)(x) > \sum (F(\{\lambda\})(x): \lambda \in A \cap S_x)$$

We would then have

$$\begin{aligned} t(x) &= F(A \cup A^c)(x) = F(A)(x) + F(A^c)(x) \\ &> \sum (F(\{\lambda\})(x): \lambda \in A \cap S_x) + \sum (F(\{\lambda\})(x): \lambda \in A^c \cap S_x) = t(x) \end{aligned}$$

which is a contradiction. We conclude that

$$F(A)(x) = \sum (F(\{\lambda\})(x): \lambda \in A \cap S_x)$$

so (3.1) holds. ■

We now give a connection between effects and observables as defined here and Hilbert space effects and observables. For $t \in \mathcal{T}$, form the Hilbert space with its usual inner product

$$H_t = \{\psi: S(t) \to \mathbb{C}: \sum |\psi(x)|^2 < \infty\}$$

For $f \in \mathcal{E}_t$, define the bounded linear operator $\hat{f}$ on H_t by

$$(\hat{f}\psi)(x) = \frac{f(x)}{t(x)} \psi(x)$$

and for an observable F satisfying $F(\mathbb{R}) = t$ define $\hat{F}(A) = [F(A)]^\wedge$ for all $A \in \mathcal{B}(\mathbb{R})$. It is clear that $\hat{f}$ is a positive operator satisfying $0 \leq \hat{f} \leq I$. Such operators are *Hilbert space effects* (Cattaneo and Giuntini, 1995; Foulis and Bennett, 1994; Greechie and Foulis, 1995). We shall show that $\hat{F}$ is a *positive operator-valued* (POV) measure and these are *Hilbert space observables.*

Theorem 3.4. (a) $f \in \mathscr{E}_t$ is sharp if and only if $\hat{f}$ is a projection. (b) If $F(\mathbb{R}) = t$, then $\hat{F}$ is a POV measure on H_t.

Proof. (a) If $f \in \mathscr{E}_t$ is sharp, then

$$(\hat{f}\psi)(x) = I_{S(f)}(x)\psi(x)$$

so $\hat{f}$ is a projection. Conversely, if $\hat{f}$ is a projection, then $\hat{f}^2 = \hat{f}$, so that $(f/t)^2 = f/t$. This implies that $f(x)/t(x)$ is 0 or 1 for every $x \in S(t)$. Hence, $f(x) = t(x)$ for every $x \in S(f)$, so f is sharp.

(b) Since $\hat{F}(\mathbb{R}) = \hat{t} = I$, $\hat{F}$ is normalized. Suppose that $A_i \in \mathcal{B}(\mathbb{R})$ are mutually disjoint, $i = 1, 2, \ldots$. Then for any $\psi \in H_t$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}
a_n &= \left\| \sum_{i=1}^{n} \hat{F}(A_i)\psi - \hat{F}\left(\bigcup_{i=1}^{\infty} A_i\right)\psi \right\|^2 \\
&= \sum_{x \in S(t)} \left| \left[\sum_{i=1}^{n} F(A_i)^\wedge(x) - F\left(\bigcup_{i=1}^{\infty} A_i\right)^\wedge(x) \right] \psi(x) \right|^2 \\
&= \sum_{x \in S(t)} \frac{1}{t(x)^2} \left| \sum_{i=1}^{n} F(A_i)(x) - F\left(\bigcup_{i=1}^{\infty} A_i\right)(x) \right|^2 |\psi(x)|^2 \\
&= \sum_{x \in S(t)} \frac{1}{t(x)^2} \left| F\left(\bigcup_{i=n+1}^{\infty} A_i\right)(x) \right|^2 |\psi(x)|^2
\end{aligned}$$

Since $\psi \in H_t$, there exist a sequence $x_j \in S(t)$ such that $\psi(x) = 0$ for $x \neq x_j$, $j = 1, 2, \ldots$, and $\|\psi\|^2 = \Sigma|\psi(x_j)|^2$. Given an $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\Sigma_{j=N+1}^{\infty}|\psi(x_j)|^2 \leq \epsilon$.

Hence,

$$\begin{aligned}
a_n &= \sum_{j=1}^{\infty} \frac{1}{t(x_j)^2} \left| F\left(\bigcup_{i=n+1}^{\infty} A_i\right)(x_j) \right|^2 |\psi(x_j)|^2 \\
&\leq \epsilon + \sum_{j=1}^{N} \frac{1}{t(x_j)^2} \left| F\left(\bigcup_{i=n+1}^{\infty} A_i\right)(x_j) \right|^2 |\psi(x_j)|^2
\end{aligned}$$

Since $F(A_i)(x_j) \in \mathbb{N}_0$ and

$$F\left(\bigcup_{i=1}^{\infty} A_i\right)(x_j) = \sum_{i=1}^{\infty} F(A_i)(x_j)$$

there exists an $M \in \mathbb{N}$ such that $n \geq M$ implies that

$$F\left(\bigcup_{i=n+1}^{\infty} A_i\right)(x_j) = 0, \qquad j = 1, \ldots, N$$

Hence, $a_n \leq \epsilon$ for $n \geq M$, so that $\lim a_n = 0$. It follows that $\hat{F}(\cup A_i) = \Sigma \hat{F}(A_i)$, where the summation converges in the strong operator topology. ■

Notice that Theorem 3.4 has limited scope. First, it can only be applied to compatible observables. Second, there are Hilbert space observables on H_t that do not have the form $\hat{F}$. For example, Theorem 3.3 shows that $\hat{F}$ must have pure point spectrum. Moreover, effects in the range of $\hat{F}$ commute and this need not hold for a general Hilbert space observable.

4. SPECTRA OF OBSERVABLES

We begin with a spectral mapping theorem for $\sigma_p(F)$.

Theorem 4.1. If F is an observable and u a Borel function, then $\sigma_p(u(F)) = u(\sigma_p(F))$.

Proof. If $\lambda \in \sigma_p(u(F))$, then $u(F)(\{\lambda\}) \neq f_0$, so that $F(u^{-1}(\{\lambda\})) \neq f_0$. Applying Theorem 3.3, there exists an $\alpha \in u^{-1}(\{\lambda\}) \cap \sigma_p(F)$. Since $\alpha \in \sigma_p(F)$ and $u(\alpha) = \lambda$, we have $\lambda \in u(\sigma_p(F))$. Conversely, suppose that $\lambda \in u(\sigma_p(F))$. Then there exists an $\alpha \in \sigma_p(F)$ such that $\lambda = u(\alpha)$. Hence,

$$u(F)(\{\lambda\}) = F(u^{-1}(\{\lambda\})) \geq F(\{\alpha\}) \neq f_0$$

so that $\lambda \in \sigma_p(u(F))$. We conclude that $\sigma_p(u(F)) = u(\sigma_p(F))$. ■

Lemma 4.2. If F is an observable, then there exists a largest open set A such that $F(A) = f_0$.

Proof. Let A_δ, $\delta \in \Delta$, be the collection of all open sets such that $F(A_\delta) = f_0$ and let $A = \cup A_\delta$. [Notice that this collection is nonempty because $F(\varnothing) = f_0$.] Since the topology of $\mathbb{R}$ is second countable, there exists a countable collection of open sets B_i, $i \in \mathbb{N}$, such that $A = \cup B_i$ and for every $i \in \mathbb{N}$ there exists a $\delta \in \Delta$ such that $B_i \subseteq A_\delta$. Since $F(B_i) = f_0$ for every $i \in \mathbb{N}$ and, as is easily seen, F is subadditive, we have $F(\cup_{i=1}^n B_i) = f_0$ for every $n \in \mathbb{N}$. Applying Lemma 3.1(d), we conclude that

$$F(A) = \lim_{n \to \infty} F\left(\bigcup_{i=1}^{n} B_i\right) = f_0$$

Hence, A is the largest open set such that $F(A) = f_0$. ■

The complement A^c of the set A in Lemma 4.2 is called the *spectrum* of F and is denoted by $\sigma(F)$. If $F(\mathbb{R}) = t$, it is clear that $\sigma(F)$ is the smallest closed set B such that $F(B) = t$. Even if $S(t)$ is countable, $\sigma(F)$ need not be countable. For example, let $S(t) = \{x_1, x_2, \ldots\}$ be a countable set and let $Q = \{\lambda_1, \lambda_2, \ldots\}$ be the rationals. Define $F(\{\lambda_i\}) = t(x_i)I_{x_i}$ and for $B \in \mathcal{B}(\mathbb{R})$

$$F(B) = \sum_{\lambda_i \in B} F(\{\lambda_i\})$$

Then $F(A) \neq f_0$ for any open set A, so that $\sigma(F) = \mathbb{R}$.

Lemma 4.3. A number λ is in $\sigma(F)$ if and only if for every open set A with $\lambda \in A$ we have $F(A) \neq f_0$.

Proof. Suppose $\lambda \in \sigma(F)$ and there exists an open set A with $\lambda \in A$ and $F(A) = f_0$. Then $F(A^c) = F(\mathbb{R}) = t$, so that

$$F(\sigma(F) \cap A^c) = F(\sigma(E)) + F(A^c) - F(\sigma(F) \cup A^c) = t$$

But $\sigma(F) \cap A^c$ is closed and $\lambda \notin \sigma(F) \cap A^c$, so $\sigma(F) \cap A^c$ is a proper subset of $\sigma(F)$. This is a contradiction. Conversely, suppose that $\lambda \notin \sigma(F)$. Since $\sigma(F)$ is closed, there exists an open set A with $\lambda \in A$ and $A \subseteq \sigma(F)^c$. Since $F(\sigma(F)^c) = f_0$, we have $F(A) = f_0$. ∎

The next result gives the relationship between $\sigma_p(F)$ and $\sigma(F)$ and extends the spectral mapping theorem to $\sigma(F)$. The closure of a set A is denoted by $\overline{A}$.

Theorem 4.4. (a) $\sigma(F) = \overline{\sigma_p(F)}$. (b) $\sigma(u(F)) \subseteq \overline{u(\sigma(F))}$. (c) if u is continuous, then $\sigma(u(F)) = \overline{u(\sigma(F))}$. (d) If $\sigma(F)$ is bounded and u is continuous on $\sigma(F)$, then $\sigma(u(F)) = u(\sigma(F))$.

Proof. (a) If $\lambda \in \sigma_p(F)$, then for any open set A with $\lambda \in A$ we have $F(A) \neq f_0$. By Lemma 4.3, $\lambda \in \sigma(F)$, so that $\sigma_p(F) \subseteq \sigma(F)$. Since $\sigma(F)$ is closed, we have $\overline{\sigma_p(F)} \subseteq \sigma(F)$. If $\lambda \in \sigma(F)$, then by Lemma 4.3, for any open set A with $\lambda \in A$ we have $F(A) \neq f_0$. Applying Theorem 3.3, we have $A \cap \sigma_p(F) \neq \emptyset$. Hence, $\lambda \in \overline{\sigma_p(F)}$, so that $\sigma(F) \subseteq \overline{\sigma_p(F)}$. (b) By Theorem 4.1 and (a) we have

$$\sigma(u(F)) = \overline{\sigma_p(u(F))} = \overline{u(\sigma_p(F))} \subseteq \overline{u(\sigma(F))}$$

(c) If $\lambda \in \sigma(F)$, then by (a), there exists a sequence $\lambda_i \in \sigma_p(F)$ such that $\lim \lambda_i = \lambda$. Since u is continuous, $\lim u(\lambda_i) = u(\lambda)$. Hence, $u(\lambda) \in \overline{u(\sigma_p(F))}$, so that $u(\sigma(F)) \subseteq \overline{u(\sigma_p(F))}$. We then have

$$\overline{u(\sigma(F))} \subseteq \overline{u(\sigma_p(F))} = \overline{\sigma_p(u(F))} = \sigma(u(F))$$

The result now follows from (b). (d) Since $\sigma(F)$ is bounded, $\sigma(F)$ is compact.

Since u is continuous on $\sigma(F)$, $u(\sigma(F))$ is compact and hence closed. Thus, $u(\sigma(F)) = \overline{u(\sigma(F))}$ and the result follows from (c). ∎

5. UNIVERSAL OBSERVABLES

An observable F is a *universal observable* for $t \in \mathcal{T}$ if $F(\mathbb{R}) = t$ and for any observable G such that $G(\mathbb{R}) = t$ (i.e., $G \leftrightarrow F$), we have $G = u(F)$ for some Borel function u. It is clear that two universal observables for t are equivalent in the sense that each is a Borel function of the other. We shall show later that a test need not admit a universal observable. An observable F is a *maximal observable* for $t \in \mathcal{T}$ if $F(\mathbb{R}) = t$ and for any $f \in \mathscr{E}_t$ there exists an $A \in \mathscr{B}(\mathbb{R})$ such that $F(A) = f$ (i.e., the range of F is $\mathscr{E}_t$).

Lemma 5.1. Any universal observable F for t is maximal.

Proof. Given an $f \in \mathscr{E}_t$, let $\lambda_1, \lambda_2 \in \mathbb{R}$ and define the observable G by $G(\{\lambda_1\}) = f$, $G(\{\lambda_2\}) = t - f$, and for $A \in \mathscr{B}(\mathbb{R})$.

$$G(A) = \sum_{\lambda_i \in A} G(\{\lambda_1\})$$

Since F is universal for t, there exists a Borel function u such that $G = u(F)$. Hence,

$$F(u^{-1}(\{\lambda_1\})) = u(F)(\{\lambda_1\}) = G(\{\lambda_1\}) = f$$

and we conclude that F is maximal. ∎

We shall show later that the converse of Lemma 5.1 does not hold. That is, a maximal observable need not be universal.

Theorem 5.2. Let F be a maximal observable for t. Then for any $x \in S(t)$ there exists a $\lambda \in \sigma_p(F)$ such that $F(\{\lambda\}) = I_x$ and for any $\alpha \in \sigma_p(F)$ there exists a $y \in S(t)$ such that $F(\{\alpha\}) = mI_y$ for some $m \in \mathbb{N}$ with $m \leq t(y)$.

Proof. Since F is maximal, there exists a $A \in \mathscr{B}(\mathbb{R})$ such that $F(A) = I_x$. Applying Theorem 3.3, there exists a $\lambda \in A \cap \sigma_p(F)$. Since $F(\{\lambda\}) \leq F(A) = I_x$, we have $F(\{\lambda\}) = I_x$. This proves the first statement of the theorem. Again, since F is maximal, there exists an $A \in \mathscr{B}(\mathbb{R})$ such that $F(A) = t(x)I_x$. Applying Theorem 3.3, there exist $\lambda_1, \ldots, \lambda_n \in A \cap \sigma_p(F)$ such that $\Sigma F(\{\lambda_i\}) = t(x)I_x$. Now suppose $\alpha \in \sigma_p(F)$ and $F(\{\alpha\})(x) \neq 0$, $F(\{\alpha\})(y) \neq 0$, where $x \neq y$. Then in our previous notation $\alpha \neq \lambda_i$, $i = 1, \ldots, n$. Letting $B = \{\alpha, \lambda_1, \ldots, \lambda_n\}$, we have

$$F(B)(x) = F(\{\alpha\})(x) + \sum F(\{\lambda_i\})(x) = F(\{\alpha\})(x) + t(x) > t(x)$$

which is a contradiction. Hence, $F(\{\alpha\}) = mI_y$ for some $y \in S(t)$ and $m \in \mathbb{N}$ with $m \leq t(y)$. ∎

Notice that for a maximal observable F, if $\lambda \in \sigma_p(F)$, then it is not necessary for $F(\{\lambda\}) = I_x$ for some $x \in S(t)$. For example, let $X = \{x\}$ and $t(x) = 3$. Defining $F(\{\lambda_1\}) = I_x$, $F(\{\lambda_2\}) = 2I_x$ gives a maximal observable that is not universal, as the next theorem shows. We denote the cardinality of a set A by $|A|$.

Theorem 5.3. Let F be a universal observable for t. Then for any $x \in S(t)$ there exists a set $A_x \subseteq \sigma_p(F)$ with $|A_x| = t(x)$ such that $F(\{\lambda\}) = I_x$ for every $\lambda \in A_x$ and for any $\alpha \in \sigma_p(F)$ we have $F(\{\alpha\}) = I_y$ for some $y \in S(t)$.

Proof. For $x \in S(t)$, let $n = t(x)$ and let $\lambda_1 \ldots, \lambda_{n+1} \in \mathbb{R}$ be distinct. Define the observable G by $G(\{\lambda_i\}) = I_x$, $i = 1, \ldots, n$, $G(\{\lambda_{n+1}\}) = t - t(x)I_x$, and for $A \in \mathcal{B}(\mathbb{R})$

$$G(A) = \sum_{\lambda_i \in A} G(\{\lambda_i\})$$

Since F is universal, there exists a Borel function u such that $G = u(F)$. Hence,

$$F(u^{-1}(\{\lambda_i\})) = G(\{\lambda_i\}) = I_x \qquad i = 1, \ldots, n$$

Applying Theorem 3.3, there exist unique $\alpha_i \in \sigma_p(F)$ such that $\alpha_i \in u^{-1}(\{\lambda_i\})$ and $F(\{\alpha_i\}) = I_x$, $i = 1, \ldots, n$. Let $A_x = \{\alpha_1, \ldots, \alpha_n\}$ give this postulated set A_x. If $\alpha \in \sigma_p(F)$, then since F is maximal, we conclude from Theorem 5.2 that $F(\{\alpha\}) = mI_y$ for some $y \in S(t)$ and $m \in \mathbb{N}$ with $m \leq t(y)$. If $\alpha \notin A_y$ then

$$F(A_y \cup \{\alpha\}) = F(A_y) + F(\{\alpha\}) = (t(y) + m)I_y > t(y)I_y$$

Since this is a contradiction, we conclude that $\alpha \in A_y$ so that $F(\{\alpha\}) = I_y$. ∎

For $t \in \mathcal{T}$ let $\{A_x : x \in S(t)\}$ be a collection of mutually disjoint sets such that $|A_x| = t(x)$. We define the *cardinality* of t to be $|t| = |\bigcup A_x|$. Notice that $|S(t)| \leq |t|$ and if $|S(t)|$ is infinite, then $|S(t)| = |t|$.

Corollary 5.4. (a) If F is universal for t, then $|\sigma_p(F)| = |t|$ and if G is an observable with $G(\mathbb{R}) = t$, then $|\sigma_p(B)| \leq |t|$. (b) If t admits a universal observable, then $|S(t)| \leq |t| \leq |\mathbb{R}|$. (c) A test t admits a maximal observable if and only if $|S(t)| \leq |t| \leq |\mathbb{R}|$.

It is unknown whether the converse of Corollary 5.4(b) holds. However, we shall show that this converse holds if $|t|$ is countable.

Lemma 5.5. If $|t| \leq |\mathbb{N}|$ and $F(\mathbb{R}) = G(\mathbb{R}) = t$, then there exists a Borel isomorphism $u: \mathbb{R} \to \mathbb{R}$ such that $\sigma_p(u(G)) \cap \sigma_p(F) = \varnothing$.

Proof. By Corollary 5.4(a), $|\sigma_p(F)|$, $|\sigma_p(G)| \leq |\mathbb{N}|$. If $A = \sigma_p(G) \cap \sigma_p(F) = \emptyset$, we are finished, so suppose that $A \neq \emptyset$. Then there exists a $B \in \mathcal{B}(\mathbb{R})$ such that $|B| = |A|$ and

$$B \cap [\sigma_p(G) \cup \sigma_p(F)] = \emptyset$$

Let $v: A \to B$ be a bijection and extend v to $u: \mathbb{R} \to \mathbb{R}$ by defining $u(\lambda) = v^{-1}(\lambda)$ for $\lambda \in B$ and $u(\lambda) = \lambda$ for $\lambda \in (A \cup B)^c$. Then u is a Borel isomorphism and by Theorem 4.1, we have

$$\sigma_p(u(G)) = u(\sigma_p(G)) = B \cup (\sigma_p(G)\backslash A)$$

Hence, $\sigma_p(u(G)) \cap \sigma_p(F) = \emptyset$. ∎

Theorem 5.6. If $|t| \leq |\mathbb{N}|$, then t admits a universal observable F, and F is unique to within a Borel isomorphism.

Proof. Let $S(t) = \{x(1), x(2), \ldots\}$ and let $A_i \subseteq \mathbb{R}$, $i \in \mathbb{N}$, be mutually disjoint with $|A_i| = t(x(i))$. Define the observable F by $F(\{\lambda\}) = I_{x(i)}$ if $\lambda \in A_i$ and for any $A \in \mathcal{B}(\mathbb{R})$

$$F(A) = \sum_{\lambda \in A} F(\{\lambda\})$$

Then $\sigma_p(F) = \cup A_i$ and $F(\mathbb{R}) = t$. To show that F is universal for t, let G be an observable with $G(\mathbb{R}) = t$. Applying Lemma 5.5, we can assume that $\sigma_p(F) \cap \sigma_p(G) = \emptyset$. By Theorem 3.3 and Corollary 5.4(a) there exists a countable set $\{\alpha_1, \alpha_2, \ldots\} \subseteq \mathbb{R}$ such that

$$G(A) = \sum (G(\{\alpha_i\}): \alpha_i \in A)$$

for every $A \in \mathcal{B}(\mathbb{R})$. Define $u: \mathbb{R} \to \mathbb{R}$ as follows. If

$$G(\{\alpha_1\}) = \sum_{i \in I_1} n_1(i) I_{x(i)}$$

where $n_1(i) \neq 0$, $i \in I_1 \subseteq \mathbb{N}$, choose $\lambda_i^1, \ldots, \lambda_i^{n_1(i)} \in A_i$, $i \in I_1$, and let $u(\lambda_i^{j(i)}) = \alpha_1$, $i \in I_1$, $j(i) = 1, \ldots, n_1(i)$. Continue in this way for α_2, making sure that the corresponding λ_j^i are distinct from those chosen for α_1, etc. Then $u(\lambda)$ is defined for $\lambda \in \sigma_p(F)$ and for $\lambda \notin \sigma_p(F)$, we define $u(\lambda) = \lambda$. Then u is a Borel function and for every $A \in \mathcal{B}(\mathbb{R})$ we have

$$G(A) = \sum (G(\{\alpha_k\}): \alpha_k \in A) = \sum (F(\{\lambda_i^{j(i)}\}): u(\lambda_i^{j(i)}) = \alpha_k, \alpha_k \in A)$$
$$= \sum (F(\{\lambda_i^{j(i)}\}): \lambda_i^{j(i)} \in u^{-1}(A)) = F(u^{-1}(A)) = u(F)(A)$$

Hence, $G = u(F)$ and F is a universal for t.

To prove the uniqueness of F, let H be another universal observable for t. Again by Lemma 5.5, we can assume that $\sigma_p(F) \cap \sigma_p(H) = \emptyset$. By Theorem

5.3, there exist mutually disjoint sets $B_i \subseteq \mathbb{R}$, $i \in \mathbb{N}$, such that $\sigma_p(H) = \bigcup B_i$, $|B_i| = t(x(i))$, $H(\{\lambda\}) = I_{x(i)}$ if $\lambda \in B_i$ and for every $A \in \mathcal{B}(\mathbb{R})$, $H(A) = \Sigma\,(H(\{\lambda\})\colon \lambda \in A \cap \sigma_p(H)$. Let $v\colon \sigma_p(F) \to \sigma_p(H)$ be a bijection satisfying $v(A_i) = B_i$, $i \in \mathbb{N}$, and extend v to $u\colon \mathbb{R} \to \mathbb{R}$ by defining $u(\lambda) = v^{-1}(\lambda)$ for $\lambda \in \sigma_p\,(H)$ and $u(\lambda) = \lambda$ for $\lambda \in (\sigma_p(F) \cup \sigma_p(H))^c$. Then u is a Borel isomorphism and for every $A \in \mathcal{B}(\mathbb{R})$ we have

$$\begin{aligned} H(A) &= \sum (H(\{\lambda\})\colon \lambda \in A \cap \sigma_p(H)) \\ &= \sum (F(u^{-1}(\lambda))\colon u^{-1}(\lambda) \in u^{-1}(A) \cap \sigma_p(F)) \\ &= F(u^{-1}(A)) = u(F)(A) \end{aligned}$$

Hence, $H = u(F)$ and it follows that $F = u^{-1}(H)$. ∎

Corollary 5.7. If $|t| \leq |\mathbb{N}|$ and F, G are observables satisfying $F(\mathbb{R}) = G(\mathbb{R}) = t$, then there exists an observable H and Borel functions u, v such that $F = u(G)$, $G = v(H)$.

We say that an E-test space $(X, \mathcal{T})$ is separable if $|t| \leq |\mathbb{N}|$ for every $t \in \mathcal{T}$. The following corollary gives a direct reason why compatible observables are simultaneously measurable.

Corollary 5.8. Let F and G be observables on a separable E-test space $(X, \mathcal{T})$. Then $F \leftrightarrow G$ if and only if there exists an observable H and Borel functions u, v such that $F = u(H)$, $G = v(H)$.

6. ORTHOCOMPLEMENTS

This section discusses two types of orthocomplements for an E-test space $(X, \mathcal{T})$. These orthocomplements are local because they depend on the test being performed. For $f \in \mathcal{E}\,(X, \mathcal{T})$ and $t \in \mathcal{T}$, we define the *Kleene* (or *diametrical*) *orthocomplement* of f by $f^t = t - f \wedge t$. Notice that $f^t \in \mathcal{E}$ for every $f \in \mathcal{E}$.

Lemma 6.1. (a) $f_0^t = t$, $t^t = f_0$ (b) $f \leq g$ implies $g^t \leq f^t$. (c) $f^{tt} = f \wedge t$ and $f^{tt} = f$ if and only if $f \leq t$. (d) $f^{ttt} = f$. (e) $(f \wedge g)^t = f^t \vee g^t$. (f) If $f \vee g \in \mathcal{E}$ (i.e., $f \leftrightarrow g$), then $(f \vee g)^t = f^t \wedge g^t$.

Proof. (a) is clear. (b) If $f \leq g$, then $f \wedge t \leq g \wedge t$, so that $t - g \wedge t \leq t - f \wedge t$. Hence, $g^t \leq f^t$. (c) We have

$$f^{tt} = t - (t - f \wedge t) \wedge t = t - (t - f \wedge t) = f \wedge t$$

and the next statement follows directly. (d) Applying (c) gives

$$f^{ttt} = t - f \wedge t = f^t$$

(e) Since $f \wedge g \leq f, g$, we have that $(f \wedge g)^t \geq f^t, g^t$. If $h \geq f^t, g^t$, then

$$h^t \leq f^{tt} = f \wedge t \leq f$$

and similarly, $h^t \leq g$. Hence, $h^t \leq f \wedge g$, so that

$$h \geq h \wedge t = h^{tt} \geq (f \wedge g)^t$$

Hence, $(f \wedge g)^t = f^t \vee g^t$. (f) Since $f \vee g \geq f, g$, we have that $(f \vee g)^t \leq f^t$. If $h \leq f^t, g^t$, then $h^t \geq f^{tt} = f \wedge t$ and $h^t \geq g \wedge t$. Hence,

$$h^t \geq (f \wedge t) \vee (g \wedge t) = (f \vee g) \wedge t$$

Since $h \leq f^t \leq t$, we have

$$h = h \wedge t = h^{tt} \leq [(f \vee g) \wedge t]^t = (f \vee g)^t$$

Hence, $(f \vee g)^t = f^t \wedge g^t$. ■

Theorem 6.2. (a) f is sharp in t if and only if $f \leq t$ and $f \wedge f^t = f_0$. (b) f is sharp in t if and only if $f \vee f^t = t$. (c) f is globally sharp if and only if $f \wedge f^t = f_0$ for every $t \in \mathcal{T}$. (d) If $f \vee f^t = t$ for every $t \in \mathscr{E}$, then f is globally sharp.

Proof. (a) Suppose that f is sharp in t. Then $f \leq t$ and $f \wedge f^t = f \wedge (t - f^t)$. If $x \notin S(f)$, then $f(x) = 0$, so that $(f \wedge f^t)(x) = 0$. If $x \in S(f)$, then $t(x) - f(x) = 0$, so again $(f \wedge f^t)(x) = 0$. Hence, $f \wedge f^t = f_0$. Conversely, suppose that $f \leq t$ and $f \wedge f^t = f \wedge (t - f) = f_0$. If $x \in S(f)$, then $f(x) \neq 0$, so that $t(x) - f(x) = 0$. Hence, f is sharp in t. (b) If f is sharp in t, then $f \leq t$ and by (a), $f \wedge f^t = f_0$. Hence, $f \vee f^t = (f \wedge f^t)^t = t$. Conversely, suppose that $f \vee f^t = t$. Then $f \leq t$ and $f \wedge f^t = (f \vee f^t)^t = f_0$, so the result follows from (a). (c) If f is globally sharp, then $f \wedge t$ is sharp in t. Since $f^t \leq t$, we have by (a) that

$$f \wedge f^t = f \wedge (f^t \wedge t) = (f \wedge t) \wedge f^t = (f \wedge t) \wedge (f \wedge t)^t = f_0$$

Conversely, suppose that $f \wedge f^t = f_0$ for every $t \in \mathcal{T}$. Then for any $t \in \mathcal{T}$, we have

$$(f \wedge t) \wedge (f \wedge t)^t = (f \wedge t) \wedge f^t = f \wedge (f^t \wedge t) = f \wedge f^t = f_0$$

Applying (a) gives that $f \wedge t$ is sharp in t. Hence, f is globally sharp. (d) If $f \vee f^t = t$, then $f \wedge f^t = (f \vee f^t)^t = f_0$. The result now follows from (c). ■

For $f \in \mathscr{E}(X, \mathcal{T})$, $t \in \mathcal{T}$, we define the *Brouwer* (or *intuitionistic*) *orthocomplement* of f by $f_t = tI_{S(f)^c}$. Notice that $f_t \in \mathscr{E}$ for every $f \in \mathscr{E}$.

Lemma 6.3. (a) $f_{0t} = t$, $t_t = f_0$. (b) $g_t \leq f_t$ if and only if $S(t) \cap S(g)^c \subseteq S(t) \cap S(f)^c$. (c) $f \leq g$ implies $g_t \leq f_t$. (d) $(f \wedge t)_t = f_t$. (e) $f \wedge f_t = f_0$.

(f) $f_{tt} = tI_{S(f)}$. (g) $f_{tt} \geq f \wedge t$ and $f_{tt} \geq f$ if and only if $f \leq t$. (h) $f_{ttt} = f_t$. (i) $(f \wedge g)_t = f_t \vee g_t$. (j) If $f \vee g \in \mathscr{E}$ (i.e., $f \leftrightarrow g$), then $(f \vee g)_t = f_t \wedge g_t$.

Proof. (a) is clear. (b) If $g_t \leq f_t$, then $S(g_t) \subseteq S(f^t)$. Hence, $S(g)^c \subseteq S(f)^c$ and the result follows. Conversely, suppose that $S(t) \cap S(g)^c \subseteq S(t) \cap S(f)^c$. We then have

$$S(g_t) = S(t) \cap S(g)^c \subseteq S(t) \cap S(f)^c = S(f_t)$$

Hence, if $x \in S(g_t)$, we have $x \in S(f_t)$ and $g_t(x) = t(x) = f_t(x)$. Also, if $x \notin S(g_t)$, then $g_t(x) = 0 \leq f_t(x)$. We conclude that $g_t \leq f_t$. (c) If $f \leq g$, then $S(f) \subseteq S(g)$, so that $S(g)^c \subseteq S(f)^c$. The result now follows from (b). (d) Since $S(t) \cap S(f)^c = S(t) \cap S(f \wedge t)^c$, the result follows from (b). (e) If $f(x) = 0$, then $(f \wedge f_t)(x) = 0$ and if $f(x) \neq 0$, then $f_t(x) = 0$, so that $(f \wedge f_t)(x) = 0$. Hence, $f \wedge f_t = f_0$. (f) If $x \in S(f)$, then $x \in S(f_t)^c$, so that $f_{tt}(x) = t(x)$. Suppose that $x \in S(f)_c$. Then $f_t(x) = t(x)$. If $t(x) \neq 0$, then $x \in S(f_t)$, so that $f_{tt}(x) = 0$. If $t(x) = 0$, then $x \in S(f_t)^c$, so that $f_{tt}(x) = t(x) = 0$. The result now follows. (g) If $x \in S(f)$, then $f_{tt}(x) = t(x) \geq (f \wedge t)(x)$. If $x \in S(f)^c$, then $f_{tt}(x) = 0 = (f \wedge t)(x)$. Hence, $f_{tt} \geq f \wedge t$. If $f \leq t$, then $f = f \wedge t \leq f_{tt}$. The converse follows because $f_{tt} \leq t$. (h) From (c), (d), and (g) we have $f_{ttt} \leq (f \wedge t)_t = f_t$. Since $f_t \leq t$, we have from (g) that $f_{ttt} \geq f_t$. (i) Suppose that $f(x) \leq g(x)$. If $f(x) = 0$, then $f_t(x) = t(x)$, so that $(f_t \vee g_t)(x) = t(x)$ and $(f \wedge g)(x) = 0$, so that $(f \wedge g)_t(x) = t(x)$. If $f(x) \neq 0$, then $g(x) \neq 0$, so that $f_t(x) = g_t(x) = 0$. Hence, $(f_t \vee g_t)(x) = 0$ and $(f \wedge g)(x) \neq 0$, so that $(f \vee g)_t(x) = 0$. A similar result holds if $g(x) \leq f(x)$ and the result follows. (j) Since $f, g \geq f \vee g$, we have $(f \vee g)_t \leq f_t, g_t$. Suppose that $h \leq f_t, g_t$. Then $h_t \geq f_{tt} \geq f \wedge t$ and $h_t \geq g_{tt} \geq g \wedge t$. Hence,

$$h_t \geq (f \wedge t) \vee (g \wedge t) = (f \vee g) \wedge t$$

The result now follows because

$$h = h \wedge t \leq h_{tt} \leq [(f \vee g) \wedge t]_t = (f \vee g)_t \quad \blacksquare$$

Theorem 6.4. (a) $f_t \leq f^t$. (b) $f_{tt} = (f_t)^t$. (c) f_t is sharp in t. (d) f_{tt} is the smallest sharp element in $\mathscr{E}_t$ such that $f_{tt} \geq f \wedge t$. (e) $(f^t)_t$ is the largest sharp element in $\mathscr{E}_t$ such that $(f^t)_t \leq f \wedge t$. The following statements are equivalent: f is sharp in $t, f = f_{tt}, f = (f^t)_t, f \leq t$, and $f^t = f_t$. (g) The following statements are equivalent: f is globally sharp, $f_{tt} = f \wedge t$ for every $t \in \mathscr{T}$, $f^t = f_t$ for every $t \in \mathscr{T}$.

Proof. (a) If $f_t(x) = 0$, then $f_t(x) \leq f^t(x)$. If $f_t(x) \neq 0$, then $x \in S(t) \cap S(f)^c$ and $f_t(x) = t(x) = f^t(x)$. (b) Applying Lemma 6.3(f), we have

$$(f_t)^t = t - f_t = t - tI_{S(f)^c} = t(1 - I_{S(f)^c}) = tI_{S(f)} = f_{tt}$$

(c) As in (a), if $f_t(x) \neq 0$, then $f_t(x) = t(x)$. (d) By (c), f_{tt} is sharp in t and by Lemma 6.3(g), $f_{tt} \geq f \wedge t$. Suppose that h is sharp in t and $f \wedge t \leq h$. If $f_{tt}(x) \neq 0$, then by Lemma 6.3(f), $x \in S(t) \cap S(f)$. Hence, $(f \wedge t)(x) \neq 0$, so that $h(x) = t(x) = f_{tt}(x)$. Thus, $f_{tt} \leq h$ and the result follows. (e) By (a) and Lemma 6.1(c) we have $(f')_t \leq f'' = f \wedge t$ and by (c), $(f')_t$ is sharp in t. Suppose that h is sharp in t and $h \leq f \wedge t$. If $h(x) \neq 0$, then $h(x) = t(x)$, so that $(f \wedge t)(x) = t(x)$ and $f(x) = t(x)$. Hence, $f'(x) = 0$, so that

$$(f')_t(x) = t(x) = h(x)$$

Hence, $h \leq (f')_t$ and the result follows. (f) This follow from (d), (e), and the fact that f is sharp in t if and only if $f \leq t$ and $t - f = f_t$. (g) This follows from (d) and (f). ■

Denote the sharp elements in $\mathscr{E}_t$ by $\mathscr{S}_t$. The next result shows that $\mathscr{S}_t$ is a classical structure.

Corollary 6.5. $(\mathscr{S}_t, \leq, ')$ is a complete atomic Boolean algebra that is isomorphic to $2^{s(t)}$.

Proof. If $f, g \in \mathscr{S}_t$, then clearly $f \wedge g$ and $f \vee g \in \mathscr{S}_t$, so that $(\mathscr{S}_t, \leq)$ is a distributive lattice. If $f \in \mathscr{S}_t$, then by Theorem 6.4(f), $f' = f_t \in \mathscr{S}_t$. Moreover, by Lemmas 6.1 and 6.3 we have $f'' = f$, $f \leq g$ implies $g' \leq f'$, and $f \wedge f' = f_0$ for all $f, g \in \mathscr{S}_t$. Hence, $(\mathscr{S}_t, \leq, ')$ is a Boolean algebra that is clearly complete. If $f \in \mathscr{S}_i$ and $f(x) \neq 0$, then $t(x)I_x$ is an atom in $\mathscr{S}_t$ and $t(x)I_x \leq f$, so that $(\mathscr{S}_t, \leq, ')$ is atomic. It is evident that the mapping $\phi: \mathscr{S}_t \to 2^{s(t)}$ given by $\phi(f) = S(f)$ is an isomorphism. ■

7. The Operation $\oplus_t$

We now define a natural local sum on $\mathscr{S}(X, \mathscr{T})$. For $t \in \mathscr{T}$, define the binary operation $\oplus_t$: $\mathscr{E} \times \mathscr{E} \to \mathscr{E}$ by $f \oplus_t g = (f + g) \wedge t$.

Lemma 7.1. (a) $f \oplus_t g = g \oplus_t f$. (b) $f \oplus_t f_0 = f \wedge t$. (c) $f \oplus_t t = t$. (d) $f \oplus_t f' = t$. (e) $(f \wedge t) \oplus_t (g \wedge t) = f \oplus_t g$. (f) $f \oplus_t (g \oplus_t h) = (f \oplus_t g) \oplus_t h$.

Proof. (a)–(c) are clear. (d) First note that

$$f \oplus_t f' = (f + t - f \wedge t) \wedge t$$

If $f(x) \leq t(x)$ or if $f(x) > t(x)$, then the right side of the previous equation equals $t(x)$. (e) if $f(x) + g(x) \leq t(x)$, we have

$$\begin{aligned}(f \oplus_t g)(x) &= [f(x) + g(x)] \wedge t(x) = f(x) + g(x)\\ &= f(x) \wedge t(x) + g(x) \wedge t(x)\\ &= [(f \wedge t + g \wedge t) \wedge t](x) = [(f \wedge t) \oplus_t (g \wedge t)](x)\end{aligned}$$

Now suppose that $f(x) + g(x) > t(x)$. Then $(f \oplus_t g)(x) = t(x)$.

Case 1. If $f(x) \geq t(x)$ or $g(x) \geq t(x)$, we have $(f \wedge t + g \wedge t)(x) \geq t(x)$, so that

$$[(f \wedge t) \oplus_t (g \wedge t)](x) = t(x)$$

Case 2. If $f(x) < t(x)$ and $g(x) < t(x)$, then $(f \wedge t + g \wedge t)(x) = f(x) + g(x)$, so again the equation in Case 1 holds. (f) Applying (e), we have

$$\begin{aligned} f \oplus_t (g \oplus_t h) &= [f + (g + h) \wedge t] \wedge t = [f \wedge t + (g + h) \wedge t] \wedge t \\ &= (f + g + h) \wedge \mathrm{t} = [(f + g) \wedge t + h \wedge t] \wedge t \\ &= [(f + g) \wedge t + h] \wedge t = (f \oplus_t g) \oplus_t h \quad \blacksquare \end{aligned}$$

Examples can be given which show that the multivalued logic condition (Chang, 1957, 1958)

(MV) $(f' \oplus_t g)' \oplus_t g = (f \oplus_t g')' \oplus_t f$

the quantum condition (Giuntini, n.d.; Gudder, 1995)

(QI) $f \oplus_t g \neq t$ implies $[g \oplus_t (f \oplus_t g)']' = f$

and the quasi-linear condition (Giuntini, 1995; Gudder, 1995)

(QL) $f \oplus_t g \neq t$ implies $f < t - g$

do not hold. However, (QI) and (QL) hold pointwise.

Lemma 7.2. (a) $(f \oplus_t g)(x) \neq t(x)$ implies $[g \oplus_t (f \oplus_t g)']'(x) f(x)$. (b) $(f \oplus_t g)(x) = t(x)$ implies $[g \oplus_t (f \oplus_t g)']'(x) = g'(x)$. (c) $(f \oplus_t g)(x) \neq t(x)$ implies $f(x) < t(x) - g(x)$.

Proof. Let $h = [g \oplus_t (f \oplus_t g)']'$. We then have

$$h = [g \oplus_t (t - (f + g) \wedge t)]' = t - [g + (t - (f + g) \wedge t)] \wedge t$$

(a) If $(f \oplus_t g)(x) \neq t(x)$, then $f(x) + g(x) < t(x)$. Hence,

$$h(x) = t(x) - [g(x) + (t(x) - f(x) - g(x))] \wedge t(x) = f(x)$$

(b) If $(f \oplus_t g)(x) = t(x)$, then $f(x) + g(x) \geq t(x)$. Hence,

$$h(x) = t(x) - [g(x) + (t(x) - t(x))] \wedge t(x) = g'(x)$$

(c) If $(f \oplus_t g)(x) \neq t(x)$, then $[f(x) + g(x)] \wedge t(x) \neq t(x)$, so that $f(x) + g(x) < t(x)$. Hence, $f(x) < t(x) - g(x)$. ■

We define the dual operation (Giuntini, 1995, n.d.) $f \odot_t g = (f' \oplus_t g')'$ and the operations (Giuntini, 1995, n.d.)

$$f \sqcap_t g = (f \oplus_t g') \odot_t g$$
$$f \sqcup_t g = (f \odot_t g') \oplus_t g$$

Theorem 7.3. (a) $f \odot_t g = (f \wedge t + g \wedge t - t) \vee f_0$. (b) $f \sqcap_t g = f \wedge g \wedge t$. (c) $f \sqcup_t g = (f \vee g) \wedge t$.

Proof. (a) By definition, we have

$$f \odot_t g = t - [(t - f \wedge t) + (t - g \wedge t)] \wedge t$$

If $f(x) > t(x)$, then both expressions in the statement (a) equal $(g \wedge t)(x)$. If $g(x) > t(x)$, then both expressions equal $(f \wedge t)(x)$. Now suppose that $f(x)$, $g(x) \le t(x)$.

Case 1. If $f(x) + g(x) \ge t(x)$, then

$$[(f \wedge t + g \wedge t - t) \vee f_0](x) = f(x) + g(x) - t(x)$$

and

$$(f \odot_t g)(x) = t(x) - [2t(x) - f(x) - g(x)] \wedge t(x) = f(x) + g(x) - t(x)$$

Case 2. If $f(x) + g(x) \le t(x)$, then

$$[(f \wedge t + g \wedge t - t) \vee f_0] = 0 = (f \odot_t g)(x)$$

(b) If $g(x) \ge t(x)$, then by (a) we have

$$(f \sqcap_t g)(x) = (f \odot_t g)(x) = (f \wedge t)(x) = (f \wedge g \wedge t)(x)$$

If $g(x) \le t(x)$, then by (a) we have

$$(f \sqcap_t g)(x) = \{[f \oplus_t (t - g)] \odot_t g\}(x) = \{[(t + f - g) \wedge t] \odot_t g\}(x)$$
$$= \{[(t + f - g) \wedge t + g - t] \vee f_0\}(x)$$

Case 1. If $f(x) \le g(x)$, then $t(x) + f(x) - g(x) \le t(x)$, so that

$$(f \sqcap_t g)(x) = f(x) = (f \wedge g \wedge t)(x)$$

Case 2. If $f(x) \ge g(x)$, then $t(x) + f(x) - g(x) \ge t(x)$, so that

$$(f \sqcap_t g)(x) = g(x) = (f \wedge g \wedge t)(x)$$

(c) If $g(x) \ge t(x)$, then by (a) we have

$$(f \sqcup_t g)(x) = (g \wedge t)(x) = t(x) = [(f \vee g) \wedge t](x)$$

If $g(x) \le t(x)$, then by (a) we have

$$(f \sqcup_t g)(x) = \{[f \odot_t (t - g)] \oplus_t g\}(x) = [(f \wedge t - g) \vee f_0 + g](x)$$

Case 1. If $f(x) \le g(x)$, then

$$(f \sqcup_t g)(x) = g(x) = [(f \vee g) \wedge t](x)$$

Case 2. If $f(x) \ge g(x)$, then

$$(f \sqcup_t g)(x) = (f \wedge t)(x) = [(f \vee g) \wedge t](x) \quad \blacksquare$$

Notice that in Theorem 7.3(c), $f \vee g$ need not be in $\mathscr{E}$. However, $(f \vee g) \wedge t \in \mathscr{E}$, so the result makes sense.

Corollary 7.4. (a) $f \sqcap_t g = g \sqcap_t f$, $f \sqcup_t g = g \sqcup_t f$. (b) $(f \sqcap_t g)' = f' \sqcup_t g'$, $(f \sqcup_t g)' = f' \sqcap_t g'$. (c) $(f \odot_t g)' = f' \oplus_t g'$, $(f \oplus_t g)' = f' \odot_t g'$.

Proof. Part (a) follows from Theorem 7.3(b), (c). (b) Applying Theorem 7.3(b), (c) and Lemma 6.1(e), we have

$$(f \sqcap_t g)' = (f \wedge g \wedge t)' = (f \wedge g)' = f' \vee g' = (f' \vee g') \wedge t = f' \sqcup_t g'$$

Similarly, we have (where again $f \vee g$ need not be in $\mathscr{E}$)

$$(f \sqcup_t g)' = [(f \vee g) \wedge t]' = (f \vee g)' = f' \wedge g' = f' \wedge g' \wedge t = f' \sqcup_t g'$$

(c) Applying Lemma 6.1(c) and Lemma 7.1(e), we have

$$(f \odot_t g)' = (f' \oplus_t g')'' = (f' \oplus_t g') \wedge t = f' \oplus_t g'$$

$$(f \oplus_i g)' = (f \wedge t \oplus_t g \wedge t)' = (f'' \oplus_t g'')' = f' \odot_t g' \quad \blacksquare$$

The next result shows that the QMV axioms (Giuntini, 1995, n.d.; Gudder, 1995) hold.

Corollary 7.5. (a) $f \sqcup_t (g \sqcap_t f) = f \wedge t$. (b) $(f \sqcap_t g) \sqcap_t h = (f \sqcap_t g) \sqcap_t (g \sqcap_t h)$. (c) $f \oplus_t [g \sqcap_t (f \oplus_t h)'] = (f \oplus_t g) \sqcap_t [f \oplus_t (f \oplus_t h)']$. (d) $f \oplus_t (f' \sqcap_t g) = f \oplus_t g$. (e) $(f' \oplus_t g) \sqcup_t (g' \oplus_t f) = t$.

Proof. (a) Applying Theorem 7.3(b), (c) gives

$$f \sqcup_t (g \sqcap_t f) = f \sqcup_t (g \wedge f \wedge t) = [f \vee (g \wedge f \wedge t)] \wedge t = f \wedge t$$

(b) By Theorem 7.3(b), we have

$$(f \sqcap_t g) \sqcap_t (g \sqcap_t h) = (f \wedge g \wedge t) \wedge (g \wedge h \wedge t) = f \wedge g \wedge h \wedge t = (f \sqcap_t g) \sqcap_t h$$

(c) If $g(x) \leq (f \oplus_t h)'(x)$, then both sides of the equation evaluated at x equal $(f \oplus_t g)(x)$. If $g(x) \geq (f \oplus_t h)'(x)$, then both sides of the equation evaluated at x equal $[f \oplus_t (f \oplus_t h)]'(x)$. (d) By Theorem 7.3(b), we have

$$f \oplus_t (f' \sqcap_t g) = f \oplus_t (f' \wedge g) = (f + f' \wedge g) \wedge t$$

If $f'(x) \geq g(x)$, then the right side becomes

$$[(f + g) \wedge t](x) = (f \oplus_t g)(x)$$

If $f'(x) \leq g(x)$, then $t(x) \leq f(x) + g(x)$ and the right side becomes

$$[(f + f') \wedge t](x) = [(f + t - f \wedge t) \wedge t](x) = t(x) = (f \oplus_t g)(x)$$

(e) By Theorem 7.3(c), we have

$$\begin{aligned}(f' \oplus_t g) \sqcup_t (g' \oplus_t f) &= (t - f \wedge t + g) \wedge t \sqcup_t (t - g \wedge t + f) \wedge t \\ &= [(t - f \wedge t + g) \wedge t] \vee [(t - g \wedge t + f) \wedge t]\end{aligned}$$

If $(g \wedge t)(x) \leq (f \wedge t)(x)$, then

$$t(x) - (f \wedge t)(x) + g(x) \geq t(x) - (f \wedge t)(x) + (g \wedge t)(x) \geq t(x)$$

Hence, the right side evaluated at x equals $t(x)$. The same result holds if $(f \wedge t)(x) \geq (g \wedge t)(x)$. ■

The next two corollaries follow from our previous work.

Corollary 7.6. For any $t \in \mathcal{T}$, $(\mathcal{E}_t, \oplus_t, ', f_0)$ is an MV-algebra (Chang, 1957, 1958). That is, (a) $(f \oplus_t g) \oplus_t h = (g \oplus_t h) \oplus_t f$. (b) $f \oplus_t f_0 = f$. (c) $f \oplus f_0' = f_0'$. (d) $f'' = f$. (e) $(f' \oplus_t g)' \oplus_t g = (f \oplus_t g')' \oplus_t f$.

Corollary 7.7. For any $t \in \mathcal{T}$, $(\mathcal{E}_t, \vee, \wedge, ', {}_t, f_0)$ is a distributive BZ^{dM}-lattice (Cattaneo and Nisticò, 1989). That is, $\mathcal{E}_t$ is a distributive lattice with smallest element f_0 that satisfies: (a) $f'' = f$. (b) $(f \vee g)' = f' \wedge g'$. (c) $f \wedge f' \leq g \vee g'$. (d) $f \wedge f_{tt} = f$. (e) $(f \vee g)_t = f_t \wedge g_t$. (f) $f \wedge f_t = f_0$. (g) $(f_t)' = f_{tt}$. (h) $(f \wedge g)_t = f_t \vee g_t$.

It is easy to verify that $f_t \oplus f_{tt} = t$ and $f \oplus_t f_{tt} = f_{tt}$. These two properties and the previous two corollaries give the following result.

Corollary 7.8. For any $t \in \mathcal{T}$, $(\mathcal{E}_t, \oplus_t, ', {}_t, f_0)$ is an $MVBZ^{dM}$-algebra (Cattaneo *et al.*, n.d.).

8. THE SASAKI MAPPING

For a test $t \in \mathcal{T}$, we define the Sasaki mapping $\phi_t: \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ by

$$\phi_t(f, g) = f - f \wedge g'$$

The counterpart of ϕ_t in previous orthostructures has been quite useful and important (Bennett and Foulis, 1995).

Lemma 8.1. (a) $\phi_t(f, g) = f$ if and only if $f \wedge g' = f_0$. (b) $\phi_t(f, g) = g$ implies $g \leq f$. (c) $\phi_t(f, f_0) = f_0$ if and only if $f \leq t$. (d) $\phi_t(f, t) = f$. (e) $\phi_t(t, f) = f \wedge t$. (f) $g \leq h$ implies $\phi_t(f, g) \leq \phi_t(f, h)$.

Proof. (a) $\phi_t(f, g) = f$ if and only if $f - f \wedge g' = f$ and this is equivalent to $f \wedge g' = f_0$. (b) If $\phi_t(f, g) = g$, then $f - f \wedge g' = g$, so that $g \leq f$. (c) $\phi_t(f, f_0) = f_0$ if and only if $f = f \wedge t$ which is equivalent to $f \leq t$. (d) $\phi_t(f, t) = f - (f \wedge f_0) = f$. (e) We have

$$\phi_t(t, f) = t - (t \wedge f') = t - f' = f \wedge t$$

(f) If $g \leq h$, then $h' \leq g'$, so that $f \wedge h' \leq f \wedge g'$. Hence,

$$\phi_t(f, g) = f - f \wedge g' \leq f - f \wedge h' = \phi_t(f, h) \quad \blacksquare$$

Theorem 8.2. We have $\phi_t(f, g) = \phi_t(g, f)$ if and only if for $x \in X$ either $f(x), g(x) \leq t(x)$ or $f(x) = g(x)$.

Proof. If $f(x) = g(x)$, then, clearly, $\phi_t(f, g)(x) = \phi_t(g, f)(x)$. Suppose that $f(x), g(x) \leq t(x)$. We then have

$$\begin{aligned} \phi_t(f, g)(x) &= f(x) - f(x) \wedge [t(x) - (g \wedge t)(x)] \\ &= f(x) - f(x) \wedge [t(x) - g(x)] \end{aligned}$$

Case 1. If $f(x) + g(x) \leq t(x)$, then $\phi_t(f, g)(x) = 0$.

Case 2. If $f(x) + g(x) \geq t(x)$, then

$$\phi_t(f, g)(x) = f(x) + g(x) - t(x)$$

In either Case 1 or Case 2, we have $\phi_t(f, g)(x) = \phi_t(g, f)(x)$.

Conversely, suppose that $\phi_t(f, g) = \phi_t(g, f)$. Assume that $f(x) > t(x)$ and $g(x) \leq t(x)$. We then have

$$\phi_t(f, g)(x) = f(x) - g'(x)$$

$$\phi_t(g, f)(x) = g(x) - g(x) \wedge [t(x) - f(x) \wedge t(x)] = g(x)$$

Hence, $f(x) = g(x) + g'(x) = t(x)$, which is a contradiction. Thus, either $f(x), g(x) > t(x)$ or $f(x), g(x) \leq t(x)$. Suppose that $f(x), g(x) > t(x)$. We then have

$$\phi_t(f, g)(x) = f(x) - g'(x) = f(x)$$

$$\phi_t(g, f)(x) = g(x) - f'(x) = g(x)$$

Hence, $f(x) = g(x)$. $\blacksquare$

Corollary 8.3. (a) If $f(x) \neq g(x)$ for every $x \in X$ and $\phi_t(f, g) = \phi_t(g, f)$, then $f, g \in \mathscr{E}_t$. (b) If $f, g \in \mathscr{E}_t$, then $\phi_t(f, g) = \phi_t(g, f)$.

Lemma 8.4. For all $t \in \mathscr{T}, f \in \mathscr{E}$, we have

$$\phi_t(f, f)(x) = \begin{cases} 0 & \text{if } f(x) \leq \frac{1}{2}t(x) \\ 2f(x) - t(x) & \text{if } \frac{1}{2}t(x) \leq f(x) \leq t(x) \\ f(x) & \text{if } t(x) \leq f(x) \end{cases}$$

Proof. Simple verification. ■

We say that an E-test space $(X, \mathscr{T})$ is classical if $|\mathscr{T}| = 1$.

Theorem 8.5. The following statements are equivalent. (a) $(X, \mathscr{T})$ is classical. (b) There exists a $t \in \mathscr{T}$ such that $\phi_t(f, g) = \phi_t(g, f)$ for every $f, g \in \mathscr{E}$. (c) $f \wedge g = f_0$ implies $f \perp g$. (d) There exists a $t \in \mathscr{T}$ such that for every $f, g \in \mathscr{E}$ we have

$$\phi_t(f, g) = [f' \oplus_t (f \wedge g')]'$$

Proof. We shall show that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a) and that (a) $\Leftrightarrow$ (d). (a) $\Rightarrow$ (b) follows from Corollary 8.3(b). (b) $\Rightarrow$ (c). Suppose that $\phi_t(f, g) = \phi_t(g, f)$ for every $f, g \in \mathscr{E}$ and that $f \wedge g = f_0$. If $f(x) = g(x)$, then $f(x) = g(x) = 0$, so that $f(x) + g(x) \leq t(x)$. If $f(x) \neq g(x)$, then by Theorem 8.2, $f, g \in \mathscr{E}_t$. Since $f \wedge g = f_0$, we conclude that $f(x) + g(x) \leq t(x)$. Hence, $f \perp g$. (c) $\Rightarrow$ (a). Assume that (c) holds and $t \in \mathscr{T}$. Suppose that $S(t) \neq X$ and $x \in X \backslash S(t)$. But then $t \wedge I_x = f_0$, so that $t \perp I_x$, which is a contradiction. Hence, $S(t) = X$. Suppose that $s \in \mathscr{T}$ and define

$$A = \{x: s(x) \leq t(x)\}$$

$$B = \{x: t(x) < s(x)\}$$

Then $A \cap B = \varnothing$ and $A \cup B = X$. Let $f = sI_B$ and $g = tI_A$. Then $f, g \in \mathscr{E}$ and $f \wedge g = f_0$. Hence, $f \perp g$, so that $f + g \in \mathscr{E}$. But $s, t \leq f + g$. Indeed, if $x \in A$, then

$$(f + g)(x) = g(x) = t(x) \geq s(x)$$

and if $x \in B$, then

$$(f + g)(x) = f(x) = s(x) > t(x)$$

Hence, $s = t = f + g$ and $|\mathscr{T}| = 1$. (d) $\Rightarrow$ (a). Suppose that (d) holds and $s, t \in \mathscr{T}$. Then by Lemma 8.1(d), we have

$$s = \phi_t(s, t) = [s' \oplus_t (s \wedge t')]' = (s' \oplus_t f_0)' = s'' = s \wedge t \leq t$$

Hence, $s = t$, so that $|\mathcal{T}| = 1$. (a) $\Rightarrow$ (d). Suppose that $\mathcal{T} = \{t\}$. Then for every $f, g \in \mathcal{E}$ we have

$$[f^t \oplus_t (f \wedge g^t)]^t = t - f^t \oplus_t (f \wedge g^t) = t - [(t - f) + f \wedge g^t]$$
$$= f - f \wedge g^t = \phi_t(f, g) \quad \blacksquare$$

We say that an E-test space $(X, \mathcal{T})$ is *semiclassical* if $s \wedge t = f_0$ for every $s, t \in \mathcal{T}$ with $s \neq t$.

Theorem 8.6. The following statements are equivalent. (a) $(X, \mathcal{T})$ is semiclassical. (b) $f^t \wedge f^s = f_0$ for every $f \in \mathcal{E}$, $s, t \in \mathcal{T}$, $s \neq t$. (c) $f^{ts} = s$ for every $f \in \mathcal{E}$, $s, t \in \mathcal{E}$, $s \neq t$. (d) $t^s = s$ for every $s, t \in \mathcal{T}$, $s \neq t$. (e) $\phi_t(t, s) = f_0$ for every $s, t \in \mathcal{T}$, $s \neq t$. (f) $\phi_t(s, g) = s$ for every $g \in \mathcal{E}$, $s, t \in \mathcal{T}$, $s \neq t$.

Proof. (a) $\Rightarrow$ (b). If (a) holds, then $f^t \wedge f^s \leq t \wedge s = f_0$ whenever $s \neq t$. (b) $\Rightarrow$ (c). If (b) holds, then letting $f = f_0$ in (b) gives $t \wedge s = f_0$. Hence, $f^t \wedge s \leq t \wedge s = f_0$, so that $f^{st} = s - f^t \wedge s = s$ whenever $s \neq t$. (c) $\Rightarrow$ (d). If (c) holds, then letting $f = f_0$ in (c) gives $t^s = s$ whenever $s \neq t$. (d) $\Rightarrow$ (e). If (d) holds and $s \neq t$, then we have

$$\phi_t(t, s) = t - t \wedge s^t = t - t \wedge t = f_0$$

(e) $\Rightarrow$ (f). If (e) holds and $s \neq t$, then

$$f_0 = \phi_t(t, s) = t - t \wedge s^t$$

so that $s^t \geq t \wedge s^t = t$. Hence, $t - s \wedge t \geq t$ and we have $s \wedge t = f_0$. Hence, $s \wedge g^t = f_0$ and we conclude that

$$\phi_t(s, g) = s - s \wedge g^t = s$$

(f) $\Rightarrow$ (a). If (f) holds, then letting $g = f_0$ in (f) gives

$$s = s - s \wedge f_0^t = s - s \wedge t$$

Hence, $s \wedge t = f_0$ whenever $s \neq t$, so that $(X, \mathcal{T})$ is semiclassical. $\blacksquare$

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